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Moving Many Pebbles in A Graph Is Polynomial Time

by

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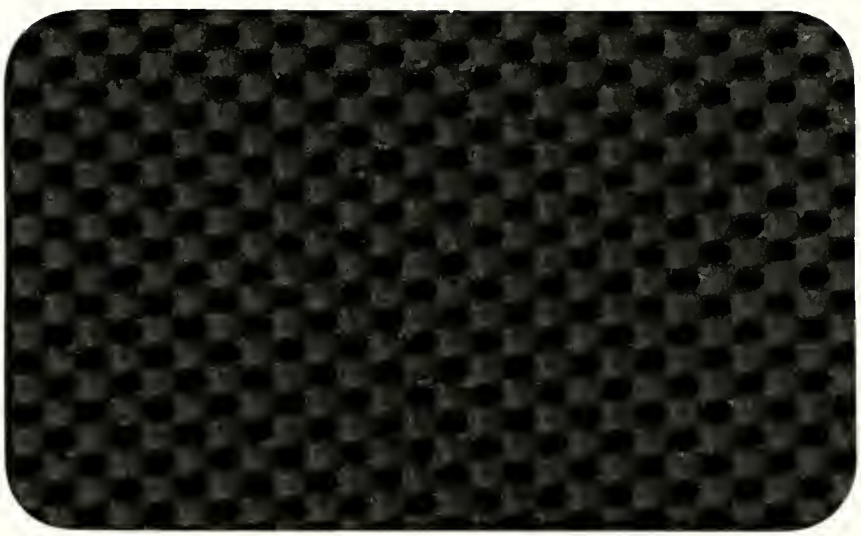

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1. Introduction

The so called 15-puzzle may be generalized to a puzzle based on an arbitrary graph. Let G be any finite simple graph of vertex set V of cardinality n + 1. By a k-pebbling we mean the placement of pebbles $1,2,\ldots,k$ $(k < n)$ on distinct vertices of G, leaving the rest of the vertices unoccupied. Formally, a k-pebbling on G is a mapping $f: V \to \{1,2,\ldots,k,\phi\}$ such that

(i) $\forall v,w \in V$, $v \neq w$ and $f(v) \neq \phi$ implies $f(v) \neq f(w)$

(ii) The set $\{x \mid \exists v \in V: f(v) = x \text{ and } x \neq \phi\} = \{1,2,\ldots,k\}$

A vertex x with $f(x) = \phi$ is called unoccupied. Two k-pebblings f,g on G are adjacent iff g can be obained from f by moving a pebble along an edge of G onto some vertex which is unoccupied in f. The relation of adjacency defined on the pebblings is symmetric and irreflexive and so it defines a new simple graph, G_m^k, with vertex set V_m^k consisting of all pebblings of G, two pebblings being joined by an edge in G_m^k iff they are adjacent. Given two pebblings f,g on G, we say that f is reachable from g (and g is reachable from f) iff there is a path in G_m^k connecting f and g. The k-pebble motion decision problem is the following: Given f,g k-pebblings on G, is g reachable from f? The k-pebble motion planning problem is the following: Given f,g

k-pebblings on G, provide a sequence of k-pebblings $h_0, h_1, \ldots, h_m$ such that $h_0 = f$, $h_m = g$ and h_{i+1} reachable from h_i for $0 \leq i \leq m-1$ if such sequence exists. Clearly the size of the input in both problems is n+k. We prove here that for a large class of graphs (including all non-separable nonbipartite graphs for any $k \leq n$ and all non-separable bipartite graphs for $k \leq n-2$ or $k = n$) both problems can be answered in polynomial time. Our work extends results of [Wilson,74] where the decision problem for $k = n$ is first posed. Our notation and terminology is heavily based on that of [Wilson, 74]. It should be immediate to note that an $O(n^{2k})$ algorithm which answers both problems can be given, based on the explicit construction of G_m^k. The space needed for this algorithm is also $O(n^{2k})$. The explicit algorithm can be made to run in $O(n^k)$ time and space if the graph G is planar.

2. The decision problem

2.a The case k = n

It is very instructive to look at the case of just one unoccupied vertex first. Let the graph θ_0 be as in Figure 1. Note that the n-labellings are now bijections so one can speak about the inverse mapping. Note also that n-labelling f,g on G are <u>adjacent</u> if $g = f^*(xy)$ where (xy) is the transposition (permutation of V(G)) which interchanges x,y and x is the vertex with $f(x) = \phi$ and y is any vertex of G adjacent to x. Thus $g(y) = \phi$ and $g(x) = f(y)$. R. Wilson [Wilson, 74] proved that:

<u>Theorem 1</u>. Let G be a finite simple nonseparable graph other than a

polygon on θ_0. Then G_m^n is connected, unless G is bipartite, in which case G_m^n has exactly two components. In this latter case, pebblings f,g on G having unoccupied vertices at even (respectively, odd) distance in G are in the same component of G_m^n if and only if $g^{-1}*f$ is and even (respectively, odd) permutation of V(G).

Corollary 1. The n-pebble motion decision problem is in P (in fact, it can be decided in $O(n^2)$ time).

Proof. Just test if G is bipartite. If not, then answer 'yes'. If G is bipartite, then find the (uniquely defined) distance of the unnocupied vertices in the initial (f) and final (g) pebblings and check for agreement with the number of inversions in $g^{-1}*f$, according to Theorem 6.1. (The case of θ_0 can be tested in constant time and the case of a polygon is testable in linear time, since, given an initial pebbling, f, of the polygon, the only reachable pebblings are the pebblings which one can get from f by moving each pebble of f clockwise or counterclockwise).

2.b. The case $k < n$.

Corollary 2. Let G be a non bipartite nonseparable graph other than a polygon or θ_0. Then it is always possible to move k pebbles from any initial to any final position.

Proof: Let f,g be the initial and final k-pebblings. Extend them (in any arbitrary way) to n-pebblings by adding fictitious pebbles. By

Theorem 1 the graph G_m^n for the n-pebblings is connected, hence the graph G_m^k for the k-pebblings is connected. •

Corollary 3. Let G be a bipartite graph, nonseparable, other than a polygon or θ_0. Let $k \leq n-2$. Then, it is always possible to move k pebbles from any initial to any final position.

Proof: Let f,g be any (initial) and (final) k-pebblings. We shall extend f,g to n-pebblings f_0, g_0 as follows: Choose $f^{-1}_0(\phi)(g^{-1}_0(\phi))$ arbitrarily, among the nonoccupied vertices in f(g). Let the distance of $f^{-1}_0(\phi)$ and $g^{-1}_0(\phi)$ be even (odd). Use the remaining 2 (or more) choices of fictitious pebbles in each of f,g to make the permutation $g^{-1}_0 * f_0$ even (odd). To do this, extend f to f_0 in any arbitrary way (except for $f^{-1}_0(\phi)$, chosen earlier). Do the same for g. If $g^{-1}_0 * f_0$ has different parity from the distance of $f^{-1}_0(\phi)$, $g^{-1}_0(\phi)$ then just swap two fictitious pebbles of g_0 to get a new final n-pebbling. Use that as your g_0. By Theorem 1 then, f_0 and g_0 belong to the same component of G_m^n. But then, f and g belong to the same component of G_m^k. •

Corollary 4. The k-pebble motion decision problem is in P for the following cases

(i) any $k \leq n$ and graph nonbipartite, polygon or θ_0

(ii) any $k < n-1$ or $k = n$ and graph bipartite

Proof: By Corollaries 1, 2 and 3. •

3. The planning problem

In the following discussion, G is a non-separable graph other than

a polygon or θ_0. We saw, in section b, how to solve the decision problem for $k \leq n$ and non-bipartite graphs or $k \in \{x: x \in N, x \leq n-2 \text{ or } x = n\}$ and bipartite graphs. In addition, in case of a positive answer to the decision problem, we saw how to extend the k-pebblings into n-pebblings in a way that a path in G_m^n from the initial to final extension implies a path in G_m^k from the initial to final k-pebblings, in the obvious way (by ignoring moves of fictitious pebbles). Hence

Corollary 3.0. A polynomial (in n) algorithm which plans the motion of n-pebblings implies a polynomial (in n) algorithm for planning the motion of k-pebblings (a) for any $k \leq n$ and a nonbipartite nonseparable graph and (b) for, $k \leq n-2$ or $k = n$, and a bipartite nonseparable graph.

3.a. Paths, permutations and groups

As in [Wilson, 74], a path p in a graph G is a sequence $p = (x_0,\ldots,x_m)$ of vertices of G such that x_{i-1} and x_i are adjacent in G, $i = 1,2,\ldots,m$. Such a path is said to be from x_0 to x_m. The path $\bar{p} = (x_m,\ldots,x_0)$ is the reverse of p. For each path $p = (x_0,\ldots,x_m)$ in G, define a permutation σ_p of V(G) as the product of transpositions

$$\sigma_p = (x_{m-1}\ x_m)*\ldots*(x_1x_2)*(x_0x_1)$$

The following fact is immediate.

Fact 1. n-labellings f,g on G are in the same component of G_m iff $f = g*\sigma_p$ for some path p in G from $f^{-1}(\phi)$ to $g^{-1}(\phi)$.

The path p of fact 1 is the path followed by the unnocupied vertex in deriving g from f.

Definition. (fix a graph G = (V,E), |V| = n+1). Let $\Gamma(x,y)$ be the set of all permuations σ_p of V where p is a path from x to y in G. We abbreviate $\Gamma(x,x)$ to be $\Gamma(x)$.

If p is a path from x to y and q a path from y to z then qp (a path from x to z) can be defined as usual. Note $\sigma_q * \sigma_p = \sigma_{qp}$ and $\sigma_p^{-1} = \sigma_{\bar{p}}$. Hence (as in [Wilson, 74] we have:

Proposition 1 For each x V(G), $\Gamma(x)$ is a group of permutations of V(G), each fixing x.

Furthermore, if p is some path from x to y in G then

$$\Gamma(x,y) = \sigma_p * \Gamma(x) = \Gamma(y) * \sigma_p$$

and

$$\Gamma(y) = \sigma_p^{-1} * \Gamma(x) * \sigma_p$$

Here it is relevant to observe that if p,q are homotopic paths (i.e. one can be derived from the other by a sequence of replacements of vertex terms y by paths (y,z,y) and the inverse of this operation) then $\sigma_p = \sigma_q$.

Definition. The order of a permutation group is the number of permutations in the group.

The degree of a permutation group is the number of distinct symbols (letters) in the permutations.

Definition. The group of all the permutations that can be performed on n symbols is called the symmetric group of degree n (and order n!)

The group of all even permutations of n symbols is called the alternating group of degree n (and order 1/2 n!). [Wilson, 73] showed the following very important theorem:

Theorem 2 [Wilson, 74]: Let G be a simple non-separable graph other than a polygon or θ_0. Then $\forall x \in V(G)$, $\Gamma(x)$ = the symmetric group on $V(G) - \{x\}$ (symb. $sym(V-\{x\})$) unlesss G is bipartite, in which case $\Gamma(x)$ = the alternating group on $V-\{x\}$ (symb. $alt(V-\{x\})$.

In fact, in view of Theorem 2 one can immediately prove Theorem 1 (which was so helpful in solving the decision problem).

Proof of Theorem 1. Let $x \in V(G)$. Each component of G_m contains an n-labelling f with $f(x) = \phi$. Given another n-labelling g (with $g(y) = \phi$) one can transform g to an n-labelling g_1 such that $g_1(x) = \phi$ (by a movement of pebbles in some path from y to x). The n-labellings f, g_1 are in the same component of G_m iff $(g_1)^{-1} * f$ is a member of $\Gamma(x)$, by definition of $\Gamma(x)$. In the case $\Gamma(x)$ is the symmetric group, all permutations are in it, so is $(g_1)^{-1} * f$, hence G_m has only one connected component.

Now, in the case $\Gamma(x)$ being the alternating group, we proceed in a slightly different way: Let $g(y) = \phi$ and $f(x) = \phi$ and let p be a path from x to y in the bipartite graph G. Then, by the definition of $\Gamma(x,y)$, $g^{-1} * f$ is in $\Gamma(x,y)$ iff f,g are in the same component of G_m. But $\Gamma(x,y) = \sigma_p * \Gamma(x)$, hence f,g are in the same component of G_m iff

$g^{-1}*f$ and σ_p have the same parity i.e. iff $g^{-1}*f$ and p have the same parity (since $\Gamma(x)$ contains all even permutations of V-x). •

To show that the n-pebble motion planning problem is in P we will

(a) Show how to decompose $g^{-1}*f$ (given f and g are in the same component of G_m) into a polynomial (in n) number of elementary permutations (The decomposition procedure has to take polynomial time).

(b) Show how to perform each elementary permutation by a polynomial length sequence of moves of simple pebbles (the algorithm which finds the sequence must be of polynomial time).

3.b An $O(n^6)$ time algorithm for motion planning of n-pebblings.

Let G be a non-separable graph (other than a polygon or θ_0) with $|V(G)| = n+1$. Let f,g be the initial and final n-pebblings, with $f(x) = \phi$, $g(y) = \phi$, $x,y \in V$.

In linear time we can derive from g an n-pebbling g_1 such that $g_1(x) = \phi$, by moving pebbles along a path p from x to y to implement $g_1 = g*\sigma_p$.

If G is bipartite, we can test if f,g belong to the same component of G_m (total time of test for G bipartite and parity of $g_1^{-1}*f$ $O(n^2)$). If so, let $h = g_1^{-1}*f$. $h \in alt(V-x)$.

If G is non-bipartite then either (i) $g_1^{-1}*f$ is in alt(x) or (ii) $g_1^{-1}*f$ is an odd permutation. Since G is non-bipartite and bisconnected, there will be closed paths p, based at x, of odd length. Find such a path (in $O(n^2)$ time) and implement $(g_1^{-1}*f)*\sigma_p$ (in linear time, by moving pebbles along p). Then $h = (g_1^{-1}*f)*\sigma_p$ belongs to alt(V-x).

We shall show how to implement the permutation h, by single pebble motions, in polynomial time. This will be enough to show that motion planning from f to g is polynomial time.

Lemma 3.b.1. Let X be any finite set, $|X| > 3$, and fix $u,v \in X$. Then the 3-cycles (uvx), $x \in X-\{u,v\}$ generate alt(X).

For proof see [Burnside, 55] or [Wilson, 74].

Lemma 3.b.2. (See [Driscoll, Furst, 83]). Let $G = \langle \mathcal{G} \rangle$ be an imprimitive group of degree n, and let $\mathcal{G}$ be a set of d-cyclic

permutations generating G where d is a constant. If $p \in G$ then a product of $O(n^2)$ generators that expresses p can be found in

$$O(|G| \cdot n^2) + O(n^6)$$

Corollary 3.b.2. A product of 3-cycles (uvx), $x \in X-\{u,v\}$, expressing $p \in alt(X)$ can be found in $O(n^6)$ time. The length of the product is $O(n)$ since the number $|G|$ of generators is $O(n)$.

Proof: By Lemmas 3.b.2 and 3.b.1 •

By using the above, we can obtain a product of at most n 3-cycles, representing h, in $O(n^6)$ time. We now show how to efficiently construct each of the 3-cycles as a composition of simple pebble motions.

Lemma 3.b.3. (A modification of a result of [Wilson, 74]).

Let $X = \{y,a_1,a_2,\ldots a_n,z,b_1,b_2,\ldots,b_m\}$ be a set of n+m+2 letters with n+m>6. Let $\pi = (a_1,a_2\ldots a_n\ z\ b_1\ldots b_m)$, $\rho = (b_1\ldots b_m\ y\ a_1\ldots a_n)$ be two cyclic permutations. Then we can get any 3-cycle of the form (yzc), $c \in X-\{y,z\}$ in polynomial time in m+n, as a composition of powers of π and ρ.

Proof: Let

$$\sigma_1 = \rho\pi^{-1}$$

$$\sigma_2 = \pi\ \sigma_1\ \pi^{-1}$$

$$\sigma_3 = \pi\ \sigma_2\ \pi^{-1}$$

$\sigma_4 = \rho^{-m}\ \sigma_1\ \rho^m$

$\sigma_5 = \sigma_2\ \sigma_4$

$\sigma_6 = \pi\ \sigma_5\ \pi^{-1}$

$\sigma_7 = \rho^{-(m-1)}\ \sigma_1\ \rho^{m-1}$

$\sigma_8 = \sigma_7\ \sigma_3$

$\sigma_9 = \sigma_8\ \sigma_6$ = a 3-cycle of the form

$(z\ a_{k+2}b_1)$ where $k = n-m$.

If $m = 0$ then $\rho\pi^{-1} = (a_1zy)$, a 3-cycle. If $m < 0$ let

$T_1 = \rho\pi^{-1}$

$T_2 = \rho^m\ T_1\ \rho^{-m}$

$T_3 = \pi^{2m}\ T_1\ \pi^{-2m}$

Let $c = \pi^{2m}a_1 \notin \{y,z,a_m,a_{m+1}\}$. Then $T_3\ T_2 = (yzc)$ a 3-cycle. The reader can verify that for $n+m > 6$ we can generate any of (yzc), $c \in X-\{y,z\}$. •

Note: The time needed to express a 3-cycle with π,ρ is $O((n+m)^2)$, if we think of π,ρ as cyclic paths and we can move pebbles around these paths.

To complete the description of the algorithm, let G be the (input) nonseparable nonpolygon graph with $v \in V(G)$ and h as before (h is in alt(V-x) and x is the unnocupied vertex). Let y,z be adjacent to x as in Figure 2. (Such $y \neq z$ always exist since G is biconnected). Then

the 3-cycles (yzc), $c \in V-\{y,z,x\}$ generate alt$(V-x)$, by Lemma 3.b.1 and can express h. Let A be the path from y to z through one such c and B the path avoiding c (such A,B are always there, by the properties of G). Assume at this point that $x \notin A$ and $x \notin B$ (at least the so called θ-graphs, which are subdivisions of the non-simple graph of Figure 3, satisfy that). Consider the paths AzB and ByA (AzB fixes y and ByA fixes z). We can implement those permutations by simple pebble motions, by moving (temporarily) the "empty" vertex to some vertex $d \notin \{y,z,c\}$ (but d on yAzB), by a motion of pebbles along dx (in linear time) and then by rotating pebbles around yAzB. (To find the paths A,B takes $O(n^2)$ time). By Lemma 3.b.3 then we can get the 3-cycle (yzc) by a polynomial in n number of single pebble motions (by $O(n^3)$ of them). This concludes the algorithm, since we showed how to move pebbles around to achieve h in $O(n^4)$ time. The total time of execution of the algorithm is bounded by the time needed to decompose h into 3-cycles i.e. $O(n^6)$.

Note: Clearly if $x \in A$ or $x \in B$ the proof is not affected.

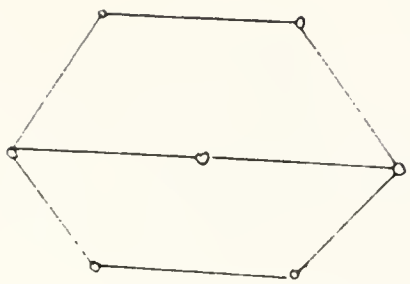
By the Corollary 3.0 we get

Theorem 3: We can solve the k-pebble motion planning problem in $O(n^6)$ time for (i) any biconnected nonbipartite graph G of n+1 vertices and any $k < n$ and (ii) any biconnected bipartite graph and $k < n$, $k \neq n-1$.

4. Discussion and further work

The case of connected (but not biconnected) graphs and $k < n$ is an open problem. (For $k = n$, the pebbles in each biconnected component cannot change component). The case of $k = n-1$ and G biconnected and bipartite is an open problem. This work was motivated by the problem of planning the motion of many objects, usually encountered in Robotics applications.

Figure 1



The graph G_0

Figure 2

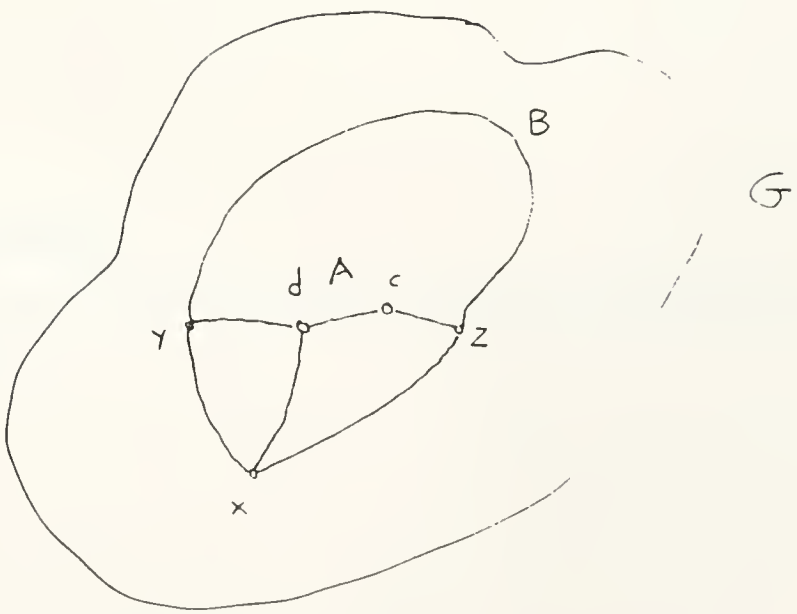
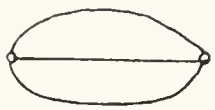


Figure 3



A θ-graph

Acknowledgements

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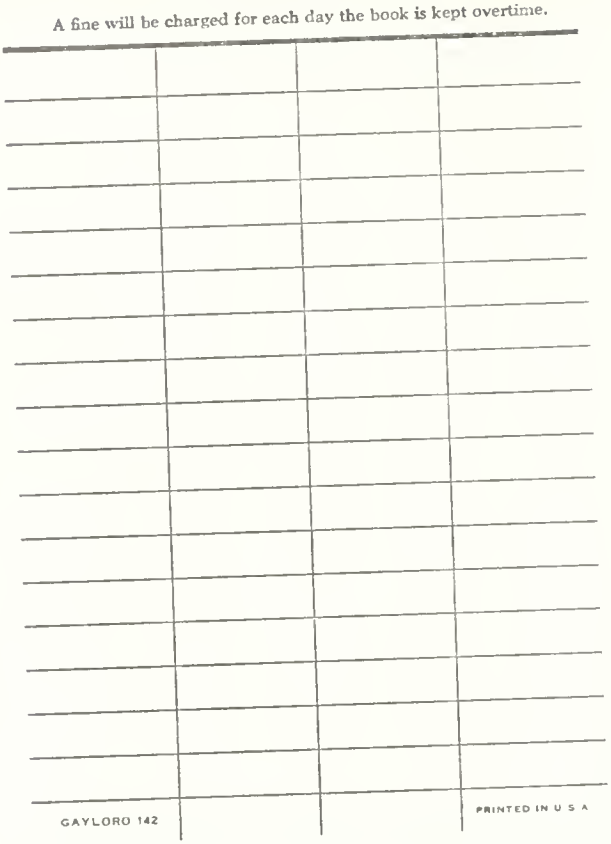

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